Onsager's algebra and superintegrability

1990 J. Phys. A: Math. Gen. 23 2245

(http://iopscience.iop.org/0305-4470/23/12/010)

J. Phys. A: Math. Gen. 23 (1990) 2245-2261. Printed in the UK

Onsager's algebra and superintegrability

Brian Davies

School of Mathematical Sciences, Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

Received 3 January 1990

Abstract. We consider the irreducible representations of the Onsager algebra, and show that for a finite system, possessing such an algebra leads to an Ising-like structure in the spectra of associated Hamiltonians and transfer matrices. The chiral Potts model is considered as an example. For transfer in the diagonal direction, it is known to be superintegrable. For transfer in the principal direction, a new superintegrable solution manifold is found.

1. Introduction

The determination of the free energy of the planar Ising model in zero magnetic field is a cornerstone in the theory of phase transitions and critical phenomena, laid by Onsager (1944). In that paper, the transfer matrix along a principal lattice direction was diagonalised and the partition function obtained. Onsager's work was simplified by Kaufman (1949) and by Schultz *et al* (1964), showing that the fermion algebra is natural for the Ising model. Important as these results are, there is much more to be found in Onsager's work. He discovered the star-triangle relation and demonstrated the existence of commuting families of transfer matrices as a consequence. Since then, Baxter (1982) has used commuting transfer matrices as a powerful tool for producing new solutions of exactly solvable models. The star-triangle relation has grown into the theory of Yang-Baxter equations (Kulish and Sklyanin 1982), also it is the foundation of the quantum inverse scattering method (Faddeev 1980, Thacker 1981).

Onsager also set up an algebra in his original paper; in fact this was the crucial step in his solution of the Ising model. Scant attention has been paid to the Onsager algebra since it played no part in the solution of the six- and eight-vertex models, nor in subsequent exact solutions of statistical mechanics models. However, it has received mention in a number of papers over the past few years. Dolan and Grady (1982) considered Hamiltonians H of the general form

$$H = A_0 + kA_1 \tag{1.1}$$

where k is a parameter and A_0 and A_1 given operators. They showed that the Dolan-Grady conditions, namely

$$[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0] \qquad [A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1] \tag{1.2}$$

are sufficient to guarantee that there is an infinite sequence of commuting operators for H. This sequence is in fact generated by an Onsager algebra. The Ising model was the only concrete realisation which they gave. Subsequently, von Gehlen and

Rittenberg (1985) considered some Z_N symmetric quantum spin chain Hamiltonians of the form (1.1), and presented strong numerical evidence that they exhibit Ising-like behaviour in their spectra. They also observed that these chains satisfy the Dolan-Grady condition. Recently, a new exactly solvable two-dimensional lattice model—the chiral Potts model—has been discovered (Baxter *et al* 1988). It has commuting families of transfer matrices, for which the commuting Hamiltonians are Z_N symmetric chains of the form (1.1). A particular special case is the superintegrable chiral Potts model (Albertini *et al* 1989, Baxter 1988), which has much Ising-like structure in its solution even though it is an N-state model. The corresponding quantum spin chain is the one investigated earlier by von Gehlen and Rittenberg (1985), and again by Albertini *et al* (1989). These investigations rely heavily on numerical computation. However, Baxter (1988) found an inversion identity—at least for the ground state sector—and used it to write exact formulae for eigenvalues in that sector. In a recent paper (Baxter 1989), Baxter has shown how these results may be used to exhibit many fascinating new features of the superintegrable chiral Potts model and of an inverse model. This work is based on the Ising-like structure of the eigenvalues.

In this paper, we will be concerned with finite-dimensional Onsager algebras, and with irreducible representations thereof. The general situation, for a Hamiltonian of the type (1.1), is that there are a number of sectors labelled by the quantum numbers of symmetries such as invariance under spin and spatial translation. The linear space on which H acts may be decomposed into invariant subspaces $\mathscr{V}$ and on each of these the restrictions of A_0 and A_1 are defined as endomorphisms of $\mathscr{V}$ (linear transformations taking $\mathscr{V}$ to $\mathscr{V}$). We will simply refer to them as operators; when necessary we will explicitly indicate the invariant subspace $\mathscr{V}$ to which they are restricted.

The plan of the paper is as follows. In section 2 we consider the Onsager algebra as a classical Lie algebra, and show that the roots have a particularly simple structure. This completely determines the form of the eigenvalues of H in any irreducible sector (as a representation of the algebra). We show in section 3 that such a sector is the direct product of n factors of dimension d_j, $1 \leq j \leq n$, and that the eigenvalues in that sector fit the general form:

$$\lambda(k) = (\alpha + \beta k) + \sum_{j=1}^{n} 4m_j\sqrt{1+k^2+k(z_j+z_j^{-1})} \qquad m_j = -s_j, -s_j+1, \ldots, s_j. \tag{1.3}$$

Here z_j, z_j^{-1} are the distinct pairs of zeros of a characteristic polynomial of degree $2n$ and $d_j = (2s_j+1)$ is the dimension of an irreducible representation of $\mathfrak{sl}(2, \mathscr{C})$ associated with the pair z_j, z_j^{-1}. The linear contribution comes from the fact that the operators A_0 and A_1 may have non-zero trace in any sector. The factor 4 comes from the 16 in the Dolan-Grady condition, and ultimately, from Onsager's choice of normalisation. Of course, it is necessary to find the constants z_j, z_j^{-1} using some information additional to the algebra, such as the inversion identity given in Baxter (1989).

In section 4 we examine transfer matrices in the principal direction which may be diagonalised using the same algebra. We show that Onsager's original formula for the eigenvalues of the Ising model transfer matrix in the principal direction extends naturally to the general case. Sections 5 and 6 are devoted to the chiral Potts model. We look at the superintegrable Z_N chain in section 5, showing in particular how the fundamental building blocks—the irreducible representations of the algebra—fit together to produce the spectrum. In section 6 we examine the chiral Potts transfer matrix in the principal direction, and show that there is a superintegrable solution manifold which is distinct from the corresponding superintegrable solution manifold for transfer matrices in the diagonal direction. Section 7 has a few concluding comments.

In presenting the more mathematical results of sections 2-4, we have chosen to set out the principal facts as theorems and proofs, in the interest of delineating the principal ideas.

2. Finite-dimensional Onsager algebra

Let $\mathscr{V}$ be a finite-dimensional complex vector space, let A_0 and A_1 be operators in $\mathscr{V}$, and let G_1 be the commutator

$$[A_1, A_0] = 4G_1. \tag{2.1}$$

Then we may define an infinite sequence of operators A_m by

$$[G_1, A_m] = 2A_{m+1} - 2A_{m-1} \tag{2.2}$$

and an infinite sequence G_m by

$$[A_m, A_0] = 4G_m. \tag{2.3}$$

The set will generate a Lie algebra using closure under the operations of addition, multiplication by complex numbers and Lie multiplication: the latter defined by the commutator. We will have an Onsager algebra $\mathfrak{A}$ if the further conditions

$$[A_l, A_m] = 4G_{l-m} \tag{2.4a}$$

$$[G_l, A_m] = 2A_{m+l} - 2A_{m-l} \tag{2.4b}$$

$$[G_l, G_m] = 0 \tag{2.4c}$$

hold. Equations (2.2) and (2.3) are a subset of these: the extra relations are an infinite set of constraints. Two such constraints come from the relations $[A_2, A_1] = [A_1, A_0] = [A_0, A_{-1}]$, giving the Dolan-Grady conditions which are therefore necessary for the existence of an Onsager algebra: they are also sufficient to guarantee that the algebra generated by (2.1) to (2.3) is an Onsager algebra (Dolan and Grady 1982).

The set A_m generates a subspace in $\mathfrak{A}$ which must be finite dimensional. That is, for some value of n there must be a linear recurrence relation, of length $(2n+1)$, namely

$$\sum_{k=-n}^{n} \alpha_k A_{k-n} = 0. \tag{2.5}$$

The algebra imposes constraints on the coefficients α_k. From (2.3) we find that the G_k also satisfy the recurrence relation (2.5). Using (2.4a) we may then obtain the more general recurrence relations

$$\sum_{k=-n}^{n} \alpha_k A_{k-l} = 0 \tag{2.6a}$$

$$\sum_{k=-n}^{n} \alpha_k G_{k-l} = 0 \tag{2.6b}$$

where now l is arbitrary. Setting $l=0$ in (2.6b) and using the fact that $G_k = -G_{-k}$, we also find the condition

$$\sum_{k=-n}^{n} \alpha_{-k} G_{k-l} = 0 \tag{2.7}$$

and this leads to the conclusion that (2.5) is unaffected by the replacement $\alpha_k \to \alpha_{-k}$. That is, we also have the recurrence relation

$$\sum_{k=-n}^{n} \alpha_{-k} A_{k-l} = 0. \tag{2.8}$$

This imposes no additional constraints on $\mathfrak{A}$ if the coefficients are either symmetric or anti-symmetric:

$$\alpha_k = \pm \alpha_{-k}. \tag{2.9}$$

This is the general situation, as we now show.

Theorem 1. If n is the minimal value for a linear recurrence relation between the A_m, then the coefficients α_k are either symmetric or anti-symmetric in k.

Proof. Assume the contrary. Then we may obtain two independent relations using the combinations $(\alpha_k \pm \alpha_{-k})$. If $\alpha_n = \pm \alpha_{-n}$ then one of these is of length $(2n-1)$, contrary to the fact that n is minimal. If $\alpha_n \neq \pm \alpha_{-n}$, then we may use these two in (2.6*b*), with $l=0$, together with $G_{-k} = -G_k$, to eliminate G_n altogether. This gives a recurrence for G_k of length $(2n-1)$, and (2.3) implies the same relation for the A_k. Again we have a contradiction. □

It is an immediate corollary that there are no minimal recurrence relations of even length.

The recurrence relation is a linear constant coefficient difference equation of order $2n$. All solutions of such equations may be expressed in terms of a fundamental solution set, determined by the characteristic polynomial $f(z)$ (Brand 1966):

$$f(z) = \sum_{k=-n}^{n} \alpha_k z^{k+n}. \tag{2.10}$$

Because of the symmetry of the coefficients, $f(z) = \pm z^{2n+1} f(1/z)$, with the sign chosen according to whether the α_k are even or odd. Consequently those zeros not equal to ± 1 occur in reciprocal pairs. Suppose that the zeros are all distinct, with none equal to ± 1. Label the zeros in pairs, i.e. the zeros are $z_{\pm j}$, $1 \leq j \leq n$, with $z_{-j} = 1/z_j$. Then the general solution of the recurrence relation may be written in terms of new operators $E_j^{\pm}$ via the following invertible linear transformation (in $\mathfrak{A}$):

$$A_m = 2 \sum_{j=1}^{n} (z_j^m E_j^+ + z_j^{-m} E_j^-). \tag{2.11}$$

The coefficients of the inversion are obtained from Lagrange interpolation polynomials (Dahlquist and Björck 1974) $f_l^{\pm}(z)$ defined as

$$f_l^+(z) = \prod_{\substack{j=1\\ j\neq l}}^{n} \left(\frac{z-z_j}{z_l - z_j}\right) \prod_{j=1}^{n} \left(\frac{z - z_{-j}}{z_l - z_{-j}}\right) \qquad f_l^-(z) = \prod_{j=1}^{n} \left(\frac{z-z_j}{z_l - z_j}\right) \prod_{\substack{j=1\\ j\neq l}}^{n} \left(\frac{z - z_{-j}}{z_l - z_{-j}}\right). \tag{2.12}$$

They have the property that

$$f_l^{\pm}(z_{\pm m}) = \delta_{lm} \qquad f_l^{\pm}(z_{\mp m}) = 0. \tag{2.13}$$

Introduce the following notation for their coefficients as polynomials in z:

$$f_l^{\pm}(z) = \sum_{k=-n}^{n-1} \beta_{l\pm,k} z^{k+n}. \tag{2.14}$$

Then it follows that the inversion of (2.11) is

$$E_l^{\pm} = \frac{z_l^{\pm n}}{2} \sum_{k=-n}^{n-1} \beta_{l\pm,k} A_k. \tag{2.15}$$

Now rewrite (2.11) in the form

$$A_m = \sum_{j=1}^{n} (z_j^m + z_j^{-m})(E_j^+ + E_j^-) + (z_j^m - z_j^{-m})(E_j^+ - E_j^-) \tag{2.16}$$

and use this in (2.3). We obtain

$$\begin{aligned} 4G_m &= [A_m, A_0] \\ &= \sum_{j=1}^{n} (z_j^m + z_j^{-m})[E_j^+ + E_j^-, A_0] + (z_j^m - z_j^{-m})[E_j^+ - E_j^-, A_0]. \end{aligned} \tag{2.17}$$

Since the G_m are antisymmetric in m, we find that $[E_j^+ + E_j^-, A_0] = 0$ and that

$$G_m = \sum_{j=1}^{n} (z_j^m - z_j^{-m}) H_j \tag{2.18}$$

where $4H_j = [E_j^+ - E_j^-, A_0]$. The inversion of (2.18) is similar to (2.15). Now we may show an important result.

Theorem 2. If the zeros of the characteristic polynomial $f(z)$ are all distinct, with none equal to ± 1, then the new generators $E_j^{\pm}$, H_j of the Onsager algebra satisfy the relations

$$[E_j^+, E_k^-] = \delta_{jk} H_k \tag{2.19a}$$

$$[H_j, E_k^{\pm}] = \pm 2\delta_{jk} E_k^{\pm}. \tag{2.19b}$$

Proof. By direct computation using (2.15), we have

$$\begin{aligned} [E_j^+, E_k^-] &= \tfrac{1}{4} \sum_{l=-n}^{n-1} \sum_{m=-n}^{n-1} \beta_{j^-,l} \beta_{k^-,m} [A_l, A_m] \\ &= \sum_{l=-n}^{n-1} \sum_{m=-n}^{n-1} \beta_{j^+,l} \beta_{k^-,m} G_{l-m} \\ &= \sum_{l=-n}^{n-1} \beta_{j^+,l} z_k^m H_k \\ &= \delta_{jk} H_k. \end{aligned} \tag{2.20}$$

This is the result (2.19a). Similar computations give the relations (2.19b). □

In this case we see by construction that $\mathfrak{A}$ is the direct sum of n copies of the simple algebra $\mathfrak{sl}(2, \mathscr{C})$. The new generators are the Cartan–Weyl basis (Humphreys 1972). The root structure is particularly simple: the roots are mutually orthogonal and the Dynkin diagram is completely disconnected.

Now consider the case of repeated zeros, with none equal to ± 1. The solution of the recurrence relation may still be written in terms of a fundamental solution set. Suppose there are n pairs of zeros, with r distinct pairs. Then (2.11) takes the more general form

$$A_m = 2 \sum_{j=1}^{n} (a_{j^+,m} E_j^+ + a_{j^-,m} E_j^-) \tag{2.21}$$

where $a_{j^+,m}$ and $a_{j^-,m}$ are fundamental solutions of the recurrence for $1 \leqslant j \leqslant n$. We choose the first r fundamental solutions as

$$a_{j^+,m} = z_j^m \qquad a_{j^-,m} = z_j^{-m} \qquad 1 \leqslant j \leqslant r \tag{2.22}$$

where z_j are the distinct zeros. Solutions for a multiple zero are z_j^m, mz_j^m, $(m)(m-1)z_j^m, \ldots$, and all except the first have a multiplicative prefactor m. Thus we have

$$A_0 = 2\sum_{j=1}^{r} (E_j^+ + E_j^-). \tag{2.23}$$

Theorem 3. If the characteristic polynomial $f(z)$ has $2r$ distinct zeros, with none equal to ± 1, then the Onsager algebra $\mathfrak{A}$ is generated by $2n + r$ operators $E_j^{\mp}$, $1 \leqslant j \leqslant n$, and H_j, $1 \leqslant j \leqslant r$. They satisfy the relations (2.19) for $1 \leqslant j \leqslant r$, while the $E_j^{\pm}$, $r < j \leqslant n$, commute with all the generators of $\mathfrak{A}$. These latter operators generate the centre of $\mathfrak{A}$.

Proof. The calculations of (2.20) may be repeated, with the same result. Once this is done, (2.23) and (2.17) show that $H_j = 0$, $j > r$. This being so, $E_j^{\mp}$, $j > r$, commute with all elements of $\mathfrak{A}$. By construction, the set $E_j^{\pm}$, $r < j \leqslant n$, generate the centre of $\mathfrak{A}$. $\square$

Finally, if there are zeros equal to ± 1, we again obtain operators in the centre of $\mathfrak{A}$. Consider the case that 1 is a zero. Then, we may label it as z_{r+1}. If -1 is not a zero then 1 has multiplicity equal to at least two, and we may take

$$a_{j^+,r+1} = 1 \qquad a_{j^-,r+1} = m. \tag{2.24}$$

Consequently, the $(r+1)$th term in the expansion of A_m is $2(E_{r+1}^+ + mE_{r+1}^-)$, and (2.23) is modified to

$$A_0 = 2\sum_{j=1}^{r} (E_j^+ + E_j^-) + 2E_{r+1}^+. \tag{2.25}$$

Now return to (2.17). The $(r+1)$th term is $2[E_{r+1}^+ + mE_{r+1}^-, A_0] = 2[E_{r+1}^+, A_0]$ and from the fact that the G_m are antisymmetric in m, we find that $[E_{r+1}^+, A_0] = 0$. Thus $H_{r+1} = 0$. A similar argument prevails if -1 is a zero. Finally, if ± 1 are a pair of zeros, then we take

$$a_{j^+,r+1} = 1 \qquad a_{j^-,r+1} = -1. \tag{2.26}$$

This pair of zeros is not reciprocal. Now, the $(r+1)$th term in the expansion of A_m is $2(E_{r+1}^+ + (-1)^m E_{r+1}^-)$, and a similar argument again gives that $H_{r+1} = 0$. Finally we have the general form of theorem 3.

Theorem 4. If the characteristic polynomial $f(z)$ has $2r$ distinct zeros, not counting ± 1, then the Onsager algebra $\mathfrak{A}$ is generated by $2n + r$ operators $E_j^{\mp}$, $1 \leqslant j \leqslant n$, and H_j, $1 \leqslant j \leqslant r$. They satisfy the relations (2.19) for $1 \leqslant j \leqslant r$, while the $E_j^{\mp}$, for $r < j \leqslant n$, commute with all the generators of $\mathfrak{A}$. These latter operators generate the centre of $\mathfrak{A}$.

3. Eigenvalues of $A_0 + kA_1$

Now we are able to address a question which is most salient for quantum Hamiltonians of the form (1.1): the general dependence of the eigenvalues on k. In the case that

A_0 and A_1 are Hermitian operators, the Hamiltonian is also Hermitian for real values of k, with real eigenvalues. In anticipation of the final result, we make the variable change

$$z_j = \exp(-\mathrm{i}\theta_j). \tag{3.1}$$

Then for the Hermitian case, the θ_j will be real. Whether or not this is the case, we will replace the combinations $(z_j \pm z_j^{-1})$ by trigonometric functions. H is a one-parameter family of operators depending continuously on k, and the eigenvalues and eigenvectors must also have continuous dependence. Apart from $k = 0$ and $k \to \pm\infty$, the dependence must also be analytic in k. The objective of this section is to prove the following theorem.

Theorem 5. The eigenvalues of the operator $H(k) = A_0 + kA_1$ are all of the form

$$\lambda(k) = (\alpha + \beta k) + \sum_{j=1}^{n} 4m_j\sqrt{1 + k^2 + 2k\cos\theta_j} \qquad m_j = -s_j, -s_j + 1, \ldots, s_j. \tag{3.2}$$

Proof. First we observe that if A_0 and A_1 have a common eigenvector x, with eigenvalues α and β respectively, then x is an eigenvector of the Hamiltonian belonging to the eigenvalue $\lambda(k) = (\alpha + \beta k)$. More generally, for any pair of eigenvalues α and β of A_0 and A_1, the intersection of the eigenspaces is a subspace of $\mathscr{V}$ which is also an invariant subspace of both A_0 and A_1 on which both operators are diagonal. Call this invariant subspace $\mathscr{V}_{\alpha\beta}$, then we may decompose $\mathscr{V}$ into the direct sum of all such subspaces, and a part $\mathscr{W}$ on which the restrictions of A_0 and A_1 have no common eigenvectors. The restriction of $\mathfrak{A}$ to $\mathscr{V}_{\alpha\beta}$ is commutative and $\mathscr{V}_{\alpha\beta}$ is an eigenspace of H with eigenvalue $(\alpha + \beta k)$. This fits the formula (3.2) with $n = 0$.

Consider now the non-commutative part. An irreducible representation has no common invariant subspaces of the operators A_0 and A_1 (by definition), and therefore, by Schur's lemma, no central elements. Let $\mathscr{W}_\gamma$ be a subspace of $\mathscr{V}$ which is a minimal invariant subspace for A_0 and A_1. The restriction of A_0 and A_1 to $\mathscr{W}_\gamma$, as matrices, generates an irreducible representation of $\mathfrak{A}$, and theorems 3 and 4 show that the characteristic polynomial of the recurrence relation for this representation has distinct zeros, none equal to ± 1. Irreducible representations of the commutation relations (2.19) are given by the standard angular momentum operators of spin s_j, with dimension $(2s_j + 1)$. That is, we will obtain irreducible representations by the homomorphism

$$\varphi(E_j^{\pm}) = J_x \pm \mathrm{i}J_y \qquad \varphi(H_j) = 2J_z \tag{3.3}$$

where J_x, J_y, J_z are the usual irreducible matrix representations of angular momentum of dimension $(2s_j + 1)$. Before we make these replacements in H itself, we must be careful to fix the origin. In the subspace $\mathscr{W}_\gamma$ the operators A_0 and A_1 will have non-zero trace (in general). In fact, the addition of a constant multiple of the identity matrix to either (or both) makes no difference to the Dolan–Grady conditions or to the generating relations for the Onsager algebra $\mathfrak{A}$. So from (2.16) the representation of H in the subspace $\mathscr{W}_\gamma$ is

$$(H)_\gamma = (\alpha + \beta k) + 2\sum_{j=1}^{n} (1 + k\cos\theta_j)\varphi(E_j^+ + E_j^-) - \mathrm{i}k\sin\theta_j\,\varphi(E_j^+ - E_j^-) \tag{3.4}$$

where the linear term fixes the trace, since the irreducible representations $\varphi(E_j^{\pm})$ of $E_j^{\pm}$ are traceless. Making the replacements (3.3), each term in the sum is a linear combination:

$$4(1+k\cos\theta_j)J_x+4k\sin\theta_j\,J_y. \tag{3.5}$$

After a rotation in the xy plane we see that (3.5) is transformed to

$$4\sqrt{1+k^2+k\cos\theta_j}\,J_x. \tag{3.6}$$

This gives the quadratic part of (3.2). In fact, substituting this back into (3.4) gives an irreducible representation, and therefore proves that the formula (3.2) is the general form for all of the eigenvalues. □

We comment that restriction to subspaces $\mathscr{W}_\gamma$ is nothing more than the usual consideration of sectors for a Hamiltonian. For many problems the sectors are determined by simple explicit symmetries of the system, such as translational invariance. However, as we shall see, there may be further symmetries due to the Dolan–Grady condition, which amount to 'hidden' symmetries.

4. Transfer matrices related to the operators A_0 and A_1

Onsager's original solution of the Ising model was for a transfer matrix in a principal direction of the lattice. This transfer matrix is the product of two non-commuting operators of dimension 2^L:

$$T=\exp(a_0A_0)\exp(a_1A_1). \tag{4.1}$$

It is clear from a later paper (Onsager 1945) that Onsager knew that transfer matrices in the diagonal direction commute with a Hamiltonian $H(k)=A_0+kA_1$. For either, the common eigenvectors can be found, given a suitable representation of the operators A_0 and A_1: finding the eigenvalues is an additional problem. For the principal direction this is not much more difficult than the construction of the eigenvectors. However, the transfer matrices in the diagonal direction are formed from products of 2×2 matrices, one for each neighbouring pair in the row, and the eigenvalues of T are much more difficult to find than the eigenvectors of $H(k)$ (Davies and Abraham 1987). For the Ising model, the operators $\exp(a_0A_0)$ and $\exp(a_1A_1)$ are those required to construct the transfer matrix in the principal direction, and it makes no difference which direction is chosen, since the Ising model is superintegrable in the whole solution manifold. This is not so for the superintegrable chiral Potts model with $N\geqslant 3$. We shall amplify this remark in section 6; however, it motivates the present section.

The Dolan–Grady condition provides a commuting Hamiltonian for transfer matrices of the form (4.1). We state the result as a theorem, in a form which employs a slightly more general transfer matrix than (4.1).

Theorem 6. If A_0 and A_1 satisfy the Dolan–Grady condition, then the transfer matrix

$$T=\exp(\lambda a_0A_0)\exp(a_1A_1)\exp[(1-\lambda)a_0A_0] \tag{4.2}$$

commutes with the operator

$$H=\coth 2a_1A_0+\coth 2a_0A_1+\frac{\sinh(2-4\lambda)a_0}{\sinh 2a_0}G_1$$
$$-\frac{\sinh 2\lambda a_0\sinh 2(1-\lambda)a_0}{\sinh 2a_0}(A_1-A_{-1}). \tag{4.3}$$

Proof. We first set $\lambda = 0$. Then the assertion is equivalent to the formula

$$\exp(a_0 A_0) H \exp(-a_0 A_0) = \exp(-a_1 A_1) H \exp(a_1 A_1). \tag{4.4}$$

Products of the form $\exp(A) B \exp(-A)$ have a well known expansion as an infinite series of nested commutators (Wilcox 1967). We use the Dolan-Grady condition and the definitions (2.4) to telescope these, giving

$$\exp(a_0 A_0) A_1 \exp(-a_0 A_0) = \cosh^2 2a_0 A_1 + \sinh^2 2a_0 A_{-1} - \sinh 4a_0 G_1. \tag{4.5}$$

A similar calculation gives

$$\exp(a_0 A_0) G_1 \exp(-a_0 A_0) = \cosh 4a_0 G_1 - \tfrac{1}{2} \sinh 4a_0 (A_1 - A_{-1}) \tag{4.6}$$

and there are corresponding results if the roles of A_0 and A_1 are interchanged:

$$\exp(-a_1 A_1) A_0 \exp(a_1 A_1) = \cosh^2 2a_1 A_0 - \sinh^2 2a_1 A_2 - \sinh 4a_1 G_1 \tag{4.7a}$$

$$\exp(-a_1 A_1) G_1 \exp(a_1 A_1) = \cosh 4a_1 G_1 + \tfrac{1}{2} \sinh 4a_1 (A_2 - A_0). \tag{4.7b}$$

The result (4.4) follows by elementary calculation. The case that λ is arbitrary is obtained by using a similarity transformation: the required Hamiltonian is $\exp(\lambda a_0 A_0) H \exp(-\lambda a_0 A_0)$. A further short calculation using (4.5) and (4.6) leads to (4.3). □

We may use this to find the eigenvalues of the transfer matrix (4.2) in any irreducible sector of $\mathfrak{A}$. We make the symmetric choice $\lambda = \tfrac{1}{2}$, as in Onsager's original paper. It only remains to carry out the algebra. The ideas may be found in Onsager's paper, and the results will be the same, but we cannot use his derivation which depends on special properties of the Pauli matrices. The Hamiltonian H is a linear combination of operators A_0, $A_{\pm 1}$, and in any irreducible sector the analogue of (3.4) is

$$\begin{aligned}(H)_\gamma = \alpha \coth 2a_1 + \beta \coth 2a_0 + 2 \sum_{j=1}^{n} (\coth 2a_1 + \cos\theta_j \coth 2a_0) \varphi (E_j^+ + E_j^-) \\ - \mathrm{i} \sin\theta_j \operatorname{cosech} 2a_0 \varphi (E_j^+ - E_j^-)\end{aligned} \tag{4.8}$$

while the analogue of (3.5) for the jth term in the sum is

$$4[(\coth 2a_1 + \cos\theta_j \coth 2a_0) J_x + (\sin\theta_j \operatorname{cosech} 2a_0) J_y]. \tag{4.9}$$

This is the generator of a rotation about an axis in the xy plane: the angle δ_j between this axis and the x axis is found, using elementary trigonometry, to be

$$\cot\delta_j = (\sinh 2a_0 \coth 2a_1 + \cos\theta_j \cosh 2a_0)/\sin\theta_j. \tag{4.10}$$

This is equivalent to ($89d$) of Onsager (1944). The representation of the transfer matrix is a direct product of matrix representations of rotations associated with each pair of zeros z_j, z_j^{-1} That is

$$(T)_\gamma = \exp(\alpha \coth 2a_1 + \beta \coth 2a_0) \prod_{j=1}^{n} \exp[2\gamma_j (\cos\delta_j J_x + \sin\delta_j J_y)]. \tag{4.11}$$

Notice that we have written $2\gamma_j$ in the exponential, since for the Ising case the matrices $2J_k$ are the Pauli matrices. We find the constants γ_j by representing the jth term as a three-dimensional rotation of the set J_x, J_y, J_z. From $\exp(\tfrac{1}{2} a_0 A_0)$ we get $\exp(2a_0 J_x)$ and from $\exp(a_1 A_1)$ we get $\exp(\mathrm{i}\theta_j J_x) \exp(4a_1 J_x) \exp(-\mathrm{i}\theta_j J_x)$. Thus we must multiply a number of 3×3 matrices, and find the eigenvalues of the product. We know that the eigenvalues must have the values 1 and $\exp[\pm 2\gamma_j]$, so the trace gives the quantity $1+2\cosh 2\gamma_j$. Carrying out the computations, we again arrive at a formula to be found in Onsager's original paper, namely

$$\cosh \gamma_j = \cosh 2a_0 \cosh 2a_1 + \cos \theta_j \sinh 2a_0 \sinh 2a_1 . \tag{4.12}$$

The eigenvalues are unaffected by a similarity transformation, so we have proved the following theorem.

Theorem 7. The eigenvalues Λ of the transfer matrix (4.2) are all of the form

$$\log \Lambda = \alpha \coth 2a_1 + \beta \coth 2a_0 + 2\sum_{j=1}^{n} m_j\gamma_j \qquad m_j = -s_j, -s_j+1, \ldots, s_j \tag{4.13}$$

with the γ_j given by (4.12).

5. Superintegrable chiral Potts model

The chiral Potts model is an N-state model, which is Z_N symmetric, and for which there is a star–triangle relation (Baxter *et al* 1988). The interactions are between adjacent N-state spins on a square lattice; the chiral property resides in the fact that these are not symmetric functions of the two spins. We give the formula for the weights in section 6. Our interest in this section is in the superintegrable case (Albertini *et al* 1989), which for $N=2$ is the Ising model. Transfer matrices are parametrised by a pair of variables: a rapidity and a temperature-like variable which we label as k (it is the k' of Baxter). For periodic boundary conditions, transfer matrices with the same value of k commute, they also commute with a common Hamiltonian. The latter generates an Onsager algebra, and we write it as

$$H = H_0 + kH_1 . \tag{5.1}$$

For a row of L sites, H_0 and H_1 act in a vector space which is the direct product of L copies of $\mathscr{C}^N : \mathscr{V} = \mathscr{C}_1^N \otimes \mathscr{C}_2^N \otimes \ldots \otimes \mathscr{C}_L^N$. They are built from operators X_l and Z_l which act non-trivially only in $\mathscr{C}_l^N$ and which satisfy the relations

$$X_l^N = I_l \qquad Z_l^N = I_l \qquad Z_lX_l = \omega X_lZ_l . \tag{5.2}$$

Using the convention that states are labelled from 0 to $N-1$, one convenient matrix representation is given by

$$(X_l)_{ij} = \delta_{i,j+1} (\mathrm{mod}\ N) \qquad (Z_l)_{ij} = \delta_{i,j}\omega^i \tag{5.3}$$

where ω is an Nth root of unity:

$$\omega = \exp(2\pi \mathrm{i}/N). \tag{5.4}$$

The formulae for H_0 and H_1 are

$$H_0 = \sum_{l=1}^{L}\sum_{m=1}^{N-1} (1-\omega^{-m})^{-1} X_l^m \tag{5.5a}$$

$$H_1 = \sum_{l=1}^{L}\sum_{m=1}^{N-1} (1-\omega^{-m})^{-1} Z_l^m Z_{l+1}^{N-m} . \tag{5.5b}$$

With periodic boundary conditions, $Z_{L+1}=Z_1$. As von Gehlen and Rittenberg (1985) observed, in the representation where X_l is diagonal, $\Sigma(1-\omega^{-m})^{-1}X_l^m$ is the usual representation for the z component of angular momentum of a system with spin $N/2$. This representation is obtained from (5.3) by the simple rule $X_l'=Z_l, Z_l'=X_l^{-1}$. In fact, it was the empirical (numerical) observation of this fact for $N=2,3,4,5$ which led them to the formula for the weights of the integrable Hamiltonian. In this representation

$$H_0=\sum_{l=1}^{L} M_l \qquad (M_l)_{ij}=\delta_{i,j}[(N-1)/2-j]. \tag{5.6}$$

Using von Gehlen and Rittenberg's representation makes it easy to check the Dolan-Grady conditions. When this is done, we find that the operators H_0 and H_1 satisfy

$$[H_1,[H_1,[H_1,H_0]]]=N^2[H_1,H_0] \qquad [H_0,[H_0,[H_0,H_1]]]=N^2[H_0,H_1]. \tag{5.7}$$

The normalising factor N^2 is quite significant. It means that we must define A_0 and A_1 by

$$A_0=4N^{-1}H_0 \qquad A_1=4N^{-1}H_1. \tag{5.8}$$

This has considerable implications. The operators H_0 and H_1 obviously commute with the spatial and spin shift operators. The eigenvalues of these operators are, respectively, $\exp(2\pi \mathrm{i}P/L)$, $0\leqslant P<L-1$, and $\exp(2\pi\mathrm{i}Q/N)$, $0\leqslant Q<N-1$; different sectors (irreducible representations of $\mathfrak{A}$) will be labelled at least by an integer pair (P, Q). However, the Dolan-Grady condition provides a further, hidden, symmetry. To see this without looking into the recursion relation (2.5), it suffices to use perturbation theory. For small k, the eigenvalues of H are close to those of H_0, and from the definition (5.6) these differ by integers. For a chain of length L, the maximum and minimum eigenvalues of H_0 differ by $(N-1)L$: in any sector of given Q, the eigenvalues differ by multiples of N, so the different number of distinct eigenvalues of H_0 is $n=[((N-1)L-Q)/N]$ where [ ] stands for integer part. However, for small k theorem 5 informs us that the eigenvalues of (A_0+kA_1), in any irreducible sector, differ by integer multiples of 4. These two pieces of information must be fitted together via (5.8). The factor $4N^{-1}$ appearing there implies that, for $N>2$, at least N distinct irreducible representations of $\mathfrak{A}$ are needed to reproduce the spectrum of H in just one sector labelled by the pair (P, Q). The integer n refers to the largest sector which must necessarily contain the ground state. This is the sector found by Baxter (1988). The Ising case is exceptional in that the two Q sectors are just even and odd parity, and there is no problem in using spin-$\frac{1}{2}$ representations of a spin-$\frac{1}{2}$ problem. The dimensions of the $P=0$ sectors, which are the sectors found by Onsager (1944), are of course 2^n. Just how many sectors are required for $N>2$, or what is their exact size, is a question which will not be addressed in this paper. However, the complications are readily seen from the case of $N=3$ with quite short chains. We give some relevant numerical data for short chains in the appendix.

For the ground-state sectors the problem of finding the characteristic polynomial $f(z)$ which determines the coefficients z_j, z_j^{-1} has been given a closed-form solution by Baxter: he defines (Baxter (1989) equation (2.11)) the functions

$$P(\zeta^N)=\zeta^{-Q}\sum_{k=0}^{N-1}\omega^{(Q+L)k}\left(\frac{\zeta^N-1}{\zeta^N-\omega^k}\right)^L \tag{5.9}$$

which are in fact polynomials of degree n in ζ^N. (We have used ζ instead of Baxter's z.) Using (2.13) of Baxter (1989) in the form

$$\zeta^{N/2}=\tan\theta/2=\frac{1-z}{1+z} \tag{5.10}$$

we recover, as the numerator, a polynomial of degree $2n$ in our variable z, which is the characteristic polynomial for the closure of the algebra in the ground state sector.

We conclude this section with the comment that (5.9) comes from an inversion identity restricted to the ground-state sector (Baxter 1988). We also note that an exact recursive algorithm for the quantities $\cos\theta_j$ (again in the ground-state sector) was given by Albertini *et al* (1989), based on exact perturbative methods together with an assumption of the form (3.2) proved above; whilst this is not a closed formula, it is more suitable for numerical computation. Moreover, Baxter *et al* (1989) recently obtained several heirarchies of inversion identities for the chiral Potts model, with no restriction to particular sectors. It should be possible to obtain information about all sectors from these, but such considerations are outside the scope of this paper.

6. Superintegrability in the principal direction

The Boltzmann weights for the chiral Potts model are defined, up to normalisation, by (Baxter *et al* 1989)

$$(w_n/w_0)=\prod_{k=1}^{n}\left(\frac{d_pb_q-a_pc_q\omega^k}{b_pd_q-c_pa_q\omega^k}\right) \tag{6.1a}$$

$$(\bar w_n/\bar w_0)=\prod_{k=1}^{n}\left(\frac{\omega a_pd_q-d_pa_q\omega^k}{c_pb_q-b_pc_q\omega^k}\right) \tag{6.1b}$$

where a_p, b_p, c_p, d_p are the standard homogeneous parameters for the chiral Potts model, restricted to lie on the curves

$$\begin{aligned} a_p^N+k'b_p^N&=kd_p^N & \qquad k'a_p^N+b_p^N&=kc_p^N\\ ka_p^N+k'c_p^N&=d_p^N & \qquad kb_p^N+k'd_p^N&=c_p^N. \end{aligned} \tag{6.2}$$

For fixed k and k', the rapidity determines a point on the curve (6.2). We will also need the Fourier transforms of the weights. Since they are defined in an asymmetrical manner (Baxter 1989) we will repeat the definitions here:

$$w_n^f=\sum_{m=0}^{N-1}\omega^{-mn}w_n \qquad \bar w_n^f=\sum_{m=0}^{N-1}\omega^{mn}\bar w_m. \tag{6.3}$$

Explicitly:

$$(w_n^f/w_0^f)=\prod_{k=1}^{n}\left(\frac{\omega a_pc_q-c_pa_q\omega^k}{d_pb_q-b_pd_q\omega^k}\right) \tag{6.4a}$$

$$(\bar w_n^f/\bar w_0^f)=\prod_{k=1}^{n}\left(\frac{c_pb_q-a_pd_q\omega^k}{b_pc_q-d_pa_q\omega^k}\right). \tag{6.4b}$$

The star-triangle relation which is satisfied by these weights leads naturally to the consideration of transfer matrices in the diagonal direction on the lattice. Suppose we draw one diagonal row of the lattice as in figure 1, and associate two spin operators

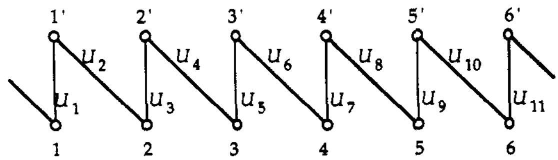

Figure 1. One (distorted) row of a lattice in the diagonal direction. The operators U_j of the interactions are indicated.

U_{2l-1}, U_{2l} with the weights, using the representation of X_l, Z_l given in (5.3). We define U_{2l-1} as

$$U_{2l-1}=\sum_{n=0}^{N-1} w_n X_l^{-n}. \tag{6.5a}$$

That is, its entries between two spins s_l, $s_l'=s_l+n$ is the Boltzmann weight w_n. The operators U_{2l} are diagonal in the chosen representation, the entries for a state with adjoining spins s_l, $s_{l+1}=s_l+n$ are $\bar{w}_n$. Thus we find

$$U_{2l}=\sum_{n=0}^{N-1} N^{-1}\left(\sum_{m=0}^{N-1}\omega^{mn}\bar{w}_m\right) Z_l^n Z_{l+1}^{N-n}. \tag{6.5b}$$

The coefficient in the braces is the Fourier transform $\bar{w}_n^f$: therefore the formula analogous to (6.5a) is

$$U_{2l}=N^{-1}\sum_{n=0}^{N-1}\bar{w}_n^f Z_l^n Z_{l+1}^{N-n}. \tag{6.5c}$$

For the diagonal direction the transfer matrix is constructed from the product $U_1U_2\ldots U_{2L-1}$, with special treatment of U_{2L} to get periodic boundary conditions. For the principal direction (figure 2) the symmetrised transfer matrix T_{princ} is given by

$$T_{\text{princ}}=(U_2U_4\ldots U_{2L})^{1/2}(U_1U_3\ldots U_{2L-1})(U_2U_4\ldots U_{2L})^{1/2}. \tag{6.6}$$

With these preliminaries complete, we consider the condition for superintegrability. For the diagonal transfer matrix the necessary condition is $x_p=y_p$, which fixes the rapidity p which is associated with the direction of transfer (Baxter *et al* 1989, section 6). The rapidity q is still free, and labels one-parameter families which commute with the Hamiltonian (5.1), (5.5). All these Hamiltonians are built from the same pair of operators H_0 and H_1 which are independent of the rapidities. However, we know from section 4 that transfer matrices of the form (4.2) for the principal direction may be constructed from H_0 and H_1 and diagonalised using the Onsager algebra. Therefore it is natural to consider two spin operators U_{2l-1}, U_{2l}, defined as

$$U_{2l-1}=\exp\left(J\sum_{k=1}^{N}(1-\omega^{-k})^{-1}X_l^k\right) \qquad U_{2l}=\exp\left(K\sum_{k=1}^{N}(1-\omega^{-k})^{-1}Z_l^kZ_{l+1}^{N-k}\right). \tag{6.7}$$

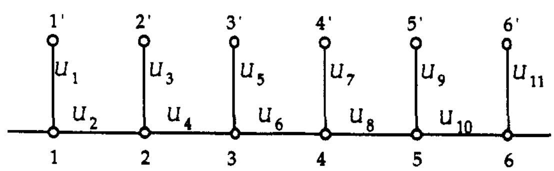

Figure 2. One row of a lattice in the principal direction. The operators U_j of the interactions are indicated.

The identity which links the forms (5.5a) and (5.6) is

$$\sum_{k=1}^{N-1} \frac{\omega^{kl}}{(1-\omega^{-k})} = \tfrac{1}{2}(N-1) - l. \tag{6.8}$$

In the representation (5.3) Z_l is diagonal: the diagonal entries of U_{2l} may be calculated using (6.8), giving the weights

$$(\bar{w}_n/\bar{w}_0) = \exp(Kn). \tag{6.9}$$

Similarly, to recover the weights from U_{2l-1} we use representation in which X_l is diagonal ($=D_{2l-1}$), given by the similarity transformation $X_l = S^{-1}Z_lS$ where the unitary matrix S has the entries $(S)_{ij} = N^{-1/2}\omega^{-ij}$. Thus $U_{2l-1} = S^{-1}D_{2l-1}S$. A simple computation gives the diagonal entries $(D_{2l-1})_{nn} = \exp(-Jn)$. Performing the similarity transformation, we get

$$(w_n/w_0) = N^{-1}\sum_{m=0}^{N-1} \omega^{mn} \exp(Jm). \tag{6.10}$$

The right-hand side of (6.10) is the inverse Fourier transform, so the analogue of (6.9) is

$$(w_n^f/w_0^f) = \exp(Jn). \tag{6.11}$$

The superintegrable solution manifold for the principal direction is defined implicitly by (6.9) and (6.11). In the standard homogeneous parameters, the condition is that the factors which occur in the products (6.1b) and (6.4a) be independent of k. The condition is the same in both cases, and is equivalent to

$$\omega a_p b_p / c_p d_p = a_q b_q / c_q d_q. \tag{6.12}$$

This is a condition which connects the two rapidities. For transfer in the principal direction, each rapidity is associated equally with the direction of transfer, so (6.12) is also intuitively correct.

7. Conclusions

The main aim of this paper has been to investigate the consequences, for a finite-lattice model in statistical mechanics, of the Onsager algebra. The connection of the Onsager algebra with Ising-like behaviour in the spectrum of a Z_N symmetric spin chain was first noted by von Gehlen and Rittenberg (1985). Albertini *et al* (1989) coined the word *superintegrable* 'because the property of possessing Onsager's operator algebra ... is clearly responsible for the extra structure'. Again, Baxter (1988), on the superintegrable model, notes that the form found for the eigenvalues is consistent with a representation, in the ground-state sector, analogous to Onsager's representation of the Ising solution. He says 'in the relevant subspace there is a similarity transformation that takes T_{row} to a direct product of 2×2 matrices ... it appears that the transformation is in fact independent of k, ... but as yet this has not been rigorously proved'. Such a proof has been provided in this paper.

As a finite-dimensional Lie algebra, the Onsager algebra has a particularly simple structure. This structure provides that a basis for an irreducible representation is a direct product of matrix representations of angular momentum operators. It does not imply that these representations must be spin-$\frac{1}{2}$. For the superintegrable chiral Potts

Table 1. Structure of irreducible blocks in the zero-momentum ($P=0$) sector, for chains of length 3, 4, 5, 6, and for $N=3$.

L	P, Q	Dim	2^n	α	β	θ_j				
3	0, 0	5	4	0.0	0.0	138.189	41.810			
			1 (×1)	0.0	0.0					
3	0, 1	3	2	−0.5	1.5	70.529				
			1 (×1)	1.0	0.0					
3	0, 2	3	2	0.5	1.5	109.471				
			1 (×1)	−1.0	0.0					
4	0, 0	8	4	−1.0	1.0	102.181	28.628			
			2 (×2)	0.5	−0.5	41.810	138.190			
4	0, 1	8	4	0.0	1.0	129.664	50.336			
			1 (×2)	0.0	1.0					
			1 (×2)	0.0	−2.0					
4	0, 2	8	4	1.0	1.0	151.362	77.819			
			2 (×2)	−0.5	−0.5	41.810	138.190			
5	0, 0	17	8	−0.5	0.5	141.669	79.295	21.247		
			2 (×3)	−0.5	0.5	234.090	115.136	53.372		
			1 (×3)	1.0	−1.0					
5	0, 1	17	8	0.5	0.5	158.753	100.705	38.331		
			2 (×3)	0.5	0.5	145.909	126.628	64.864		
			1 (×3)	−1.0	−1.0					
5	0, 2	17	4	0.0	−1.0	147.021	32.979			
			4	0.0	2.0	120.609	59.391			
			2 (×1)	1.5	0.5	70.529				
			2 (×1)	−1.5	0.5	109.471				
			1 (×4)	0.0	−1.0					
			1 (×1)	0.0	2.0					
6	0, 0	46	16	0.0	0.0	163.406	116.036	63.964	16.594	
			4	0.0	0.0	170.137	9.863			
			4	0.0	0.0	82.861	26.610			
			4	0.0	0.0	136.087	43.913			
			4	0.0	0.0	153.390	97.139			
			4	0.0	0.0	151.317	28.683			
			1 (×10)	0.0	0.0					
6	0, 1	42	8	−0.5	1.5	132.577	81.475	30.494		
			4	1.0	0.0	153.026	53.522			
			4	1.0	0.0	153.748	107.724			
			4	1.0	0.0	90.169	33.460			
			2 (×5)	−0.5	−1.5	148.299	57.716	105.637	83.329	36.544
			2 (×2)	−0.5	1.5	59.391	120.609			
			1 (×5)	1.0	0.0					
			1 (×3)	−2.0	0.0					
6	0, 2	42	8	0.5	1.5	149.506	98.525	47.423		
			4	−1.0	0.0	72.276	26.252			
			4	−1.0	0.0	126.478	26.970			
			4	−1.0	0.0	146.539	89.831			
			2 (×5)	0.5	−1.5	143.456	96.671	122.284	74.363	31.701
			2 (×2)	0.5	1.5	59.391	120.609			
			1 (×5)	−1.0	0.0					
			1 (×3)	2.0	0.0					

model the representations are in fact spin-$\frac{1}{2}$. It is interesting to speculate on the nature of superintegrable lattice models which require higher-dimensional representations. Even for the Ising case, an examination of Onsager's original paper, in which a representation of only the ground-state sector is constructed, shows that the recovery of the Ising model from its irreducible representations is a problem at least as difficult as the original solution! For the chiral Potts model, the complicated sector structure would make such a venture even more daunting.

We conclude with some comments on the superintegrability of the chiral Potts model in the principal direction. For the Ising model, which is always superintegrable, this provides an alternative framework for solution. However, for $N>2$, superintegrability is a very special property, and the two superintegrable manifolds are distinct. We are currently investigating the thermodynamics of the chiral Potts model in this new superintegrable solution manifold. This promises to be of interest, especially since the chiral Potts model is intrinsically anisotropic. The results will be published elsewhere.

Acknowledgments

I would like to thank M N Barber, R J Baxter, V V Bazhanov, A Carey, C A Hurst, B M McCoy, P A Pearce, J H H Perk and E A Scott for helpful discussions.

Appendix

We remarked at the end of section 5 on the complicated structure of the spectrum of the superintegrable spin chain: each fundamental sector labelled by a pair of quantum numbers P, Q breaks up into many irreducible representations of the Onsager algebra. For the three-state case, we have made numerical calculations for chains of length $3 \leqslant L \leqslant 6$. First we diagonalised the Hamiltonian in the relevant sector. The eigenvectors were then used to construct a basis in which the operators A_0 and A_1 are block diagonal, with each block irreducible. This being done, the irreducible blocks are, of necessity, of dimension 2^n, and the spectrum additive, given by (3.2). To within double precision in FORTRAN we checked this property for each block, and then used (3.2) to extract the quantities α, β and the angles θ_j. (For blocks of dimension 1, there is no such angle.) The information is displayed in table 1. The column labelled 'dim' is the total dimension of the sector (P, Q) and the next column shows the dimension of each irreducible block. Then follow the values of α, β and θ_j. In the case that there is more than one block of dimension 1 or 2 with the same value of α and β, we have indicated the total number of such blocks, and listed all the θ_j alongside, one for each block. The rapid proliferation is apparent from this data. It is also evident that in any sector (P, Q) all the angles θ_j are distinct, although we know of no way to prove this. This conjecture has been checked for other values of N and L, but the data is not given here as it sheds no further light on the matter.

References

Au-Yang H and Perk J H H 1989 *Proc. Taniguchi Symp.* vol 19 (New York: Kinokumiya/Academic)
Albertini G, McCoy B M, Perk J H H and Tang S 1989 *Nucl. Phys.* B **314** 741

Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
—— 1988 *Phys. Lett.* **133A** 185
—— 1989 *J. Stat. Phys.* to appear
Baxter R J, Bazhanov V V and Perk J H H 1989 *Preprint* CMA-R38-89 (Australian National University)
Baxter R J, Perk J H H and Au-Yang H 1988 *Phys. Lett.* **128A** 138
Brand L 1966 *Differential and Difference Equations* (New York: Wiley)
Dahlquist G and Björck A 1974 *Numerical Methods* (Princeton, NJ: Prentice-Hall)
Davies and Abraham 1987 *MSCR Report 20* (Australian National University)
Dolan L and Grady M 1982 *Phys. Rev.* D **25** 1587
Faddeev L D 1980 *Math. Phys. Rev.* **1** 107
Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (Berlin: Springer)
Kaufman B 1949 *Phys. Rev.* **76** 117
Kulish P P and Sklyanin E K 1982 *Integrable Quantum Field Theories* (*Lecture Notes in Physics* **151**) ed J Hietarinta and C Montonen (Berlin: Springer) p 61
Onsager L 1944 *Phys. Rev.* **65** 117
—— 1945 *Critical Phenomena in Alloys, Magnets and Superconductors* ed Mills, Ascher and Jaffe (New York: McGraw-Hill) p 3
Schultz T D, Mattis D C and Lieb E 1964 *Rev. Mod. Phys.* **36** 856
Thacker H B 1981 *Rev. Mod. Phys.* **53** 253
von Gehlen G and Rittenberg V 1985 *Nucl. Phys.* B **257[FS14]** 351
Wilcox R M 1967 *J. Math. Phys.* **8** 962